

Equilibrium of Blockchain Miners with Dynamic Asset Allocation

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Abstract— We model and analyze blockchain miners who seek to maximize the compound return of their mining businesses. The analysis of the optimal strategies finds a new equilibrium point among the miners and the mining pools, which predicts the market share of each miner or mining pool. The cost of mining determines the share of each miner or mining pool at equilibrium. We conclude that neither miners nor mining pools who seek to maximize their compound return will have a financial incentive to occupy more than 50% of the hash rate if the cost of mining is at the same level for all. However, if there is an outstandingly cost-efficient miner, then the market share of this miner may exceed 50% in the equilibrium, which can threaten the viability of the entire ecosystem.

Index Terms—Blockchain, Kelly Strategy, Equilibrium

I. INTRODUCTION

A. Background

In Bitcoin network, most of the mining power is controlled by mining pools, and most of the hash rate is produced by “mining factories” that equip ASIC mining machines [1]. Since ASIC mining machines have limited purposes other than mining, the economic behavior of a miner depends on its long-term prediction rather than a short-term benefit. Hence, it is necessary to analyze long-term incentives that the PoW mechanism gives to each miner and to establish the theory to evaluate the sustainability of the cryptoasset. The theory is useful not only for miners to determine their strategy but also for protocol designers and business entities to use permissionless blockchains for their applications. There exist prior work on the analysis of the relationship between the short-term change of mining cost and mining power [2], [3]; however, the short-term analysis does not explain how people invest in mining assets for long-term profit. We need a new approach for analyzing how the change of mining power affects profit on the long term.

B. Profitable Strategies for Repeated Games

To illustrate the difference between optimal strategies for short-term profit and those for long-term profit in repeated games, we consider the following example. Suppose that there

is a repeated game in which you can bet a certain amount and flip a coin in every stage. You gain 23% of your bet with probability 1/2, while you lose 20% of your bet with probability 1/2. Suppose that at the beginning of the game, you have 1,000 USD in your hand, and you can choose any amount of bet from your hand for each stage of the coin-flip game.

First of all, if you keep betting a constant amount, then the strategy is likely to be outperformed in the long-term by a riskfree strategy that generates the compound return of the riskfree rate. Hence betting in the game implies you will seek a better compound return rate than the riskfree rate. However, if you bet all in, then you are likely to find your money is halved after about 90 times because you are likely to result in 45 wins and 45 losses, so it will be $1000(1 + 23\%)^{45}(1 - 20\%)^{45} \approx 480$ USD.

You can improve the compound return rate by adjusting the amount of bet. Let W be the random variable that takes $W = 0.23$ with probability 1/2 and $W = -0.2$ with 1/2. Let f be a real number $0 < f < 1$ that decides the asset allocation for the bet: If we have assets A , you will bet fA in the game, and will keep $(1 - f)A$. Your assets after the coin-flip game will be $(1 - f)A + f(1 + W)A$, so the probabilistic return rate is $X = fW$. To maximize the compound return rate, you would like to maximize $E[\log(1 + X)] = \log(1 + 0.23f)0.5 + \log(1 - 0.2f)0.5$. By solving $\frac{d}{df} E[\log(1 + X)] = 0$, it is easy to see the maximum is attained when f is $f^* \approx 0.33$. Your expected log-return rate at f^* is about 0.0024, a positive rate.

The choice of optimal allocation factor f^* under the payoff function of the log-return rate is called the Kelly strategy [4], [5]. The Kelly strategy is known to be the optimal asset allocation for repeated games with respect to the expected compound return rate [6], [7].

C. Dynamic Asset Allocation in Blockchain Mining

In this paper, we apply the idea of Kelly strategy to Proof-of-Work blockchain mining. To formulate the Kelly strategy in blockchain mining, we need a model for the economics of blockchain mining that allows dynamic mining algorithms. To the best of our knowledge, the rewards of blockchain mining are modeled using the Poisson process (for example, [8]). In the model, the miners are modeled as a fixed algorithm that receives probabilistic rewards according to the Poisson process. This paper proposes a model that allows dynamic

mining algorithms by formulating the economics of mining as a binomial tree model for the probabilistic return from mining reward minus cost, while a Poisson process triggers the growth of the binomial tree. We show that the proposed model is a generalization of the existing Poisson reward model.

Using the new model, we present an analysis of the decision on dynamic asset allocation. We call the mining strategy with the optimal asset allocation the *growth-rate mining*. The analysis of the decision also finds the equilibrium point of hash rates among the growth-rate miners.

D. Predicting Share of Bitcoin Mining Pools

The equilibrium point predicts miners' shares of the hash rate of Bitcoin. Assuming all the miners have the same cost rate of mining, the growth-rate miners will occupy about 9% of the world hash rate when the world hash rate is at 85% of the break-even point. If a growth-rate miner is more cost-efficient by about 5% than other growth-rate miners, then the miner will occupy about 13%. Interestingly, the prediction coincides roughly with reality.

The growth-rate mining strategy does not threaten the security of Bitcoin with respect to the 51% attack as long as there are no miners who have outstanding cost-efficiency. However, the lower bound of the mining cost for the 51% attack is not necessarily unrealistic in the current Bitcoin environment: the equilibrium point exceeds 50% of the world hash rate if a miner is about 70% more cost-efficient than other miners.

E. Related Works

1) *Known Equilibrium Points Among Miners*: Chiu et al. claim a Nash equilibrium point from the Cournot game setting the mining reward minus cost as the payoff function for Player i [3]. In our notations defined in Section II-A, the hash rate at equilibrium is given by $\hat{M}_i = \frac{m-1}{m^2} \frac{B}{c}$ when $c_i = c$ for all i .

Pagnotta et al. and Cong et al. independently claim equilibrium points based on the CAAR utility function ([9] and [10]). Both contain an exogenous parameter that indicates the degree of risk-aversion.

2) *Mining Pools*: Wang et al. consider the mining pools' choice between being open or closed to miners: the former strategy is likely to be more efficient since attracts more miners, while the latter strategy protects the pool from certain attacks [11]. The authors model the pools' choice as a two-stage game, in which pools choose to be open or not and to attack or not, and find that weaker pools are more likely to attack.

Qin et al. study how miners select which mining pool to join, considering pay-per-share, pay-per-last- N -share, and proportional reward mechanisms [12]. The authors model pool selection as a risk decision problem based on maximum-likelihood criterion, which can provide managerial insights for miners. Liu et al. study the dynamics of mining pool selection and find that the hash rate for puzzle-solving and block propagation delay are the two major factors that determine the results of the competition between mining pools [13].

Schrijvers et al. study the incentive compatibility of mining pool reward mechanism using a game-theoretic model, in which miners can choose between reporting or delaying when they discover a share or full solution [14]. The authors show that proportional rewards are not incentive compatible, but the pay-per-last- N -shares mechanism is in a more general model, and they introduce a novel incentive compatible mechanism.

II. MINING WITH DYNAMIC ASSET ALLOCATION

A. Model

We model the economics of blockchain mining as a repeated reward process.

Environment

- B is the reward for mining the next block.
- τ is the average time interval between new block arrivals.
- r is the riskfree rate for time interval τ , usually the interest rate for the national treasury bond.

Players There are a finite number of Players. Each Player i has the following parameters.

- A set of balance sheets $\mathcal{O}_i \subset \mathcal{O}$ which the Player chooses one from. \mathcal{O} is the set of all possible balance sheets $\mathcal{O} = \{(E, L, M, F) \in \mathbb{R}_{\geq 0}^4\}$ that satisfy $E + L = M + F$, and $L = 0$ if $F \neq 0$. We call E Equity, L Liabilities, M Mining Assets, and F Riskfree Assets.
- Price of facility d_i , which is the average price for facilities that produce the unit hash rate.
- Cost rate c_i , which is the average cost for running the device per the unit hash rate per time interval of τ .

Shared information The parameters for the Environment are publicly known. Also, each Player's existence and its hash rate of M_i/d_i are publicly known.

Nakamoto Reward Process The Nakamoto Reward Process is a timeless random event for Players with balance sheets. Of all Players who play the game, Player i with Mining Assets M_i is exclusively randomly chosen with probability

$$p_i = \frac{M_i/d_i}{\sum_j M_j/d_j},$$

and obtains revenue of B . In addition, each Player j always pays cost $c_j M_j/d_j$. We call p_i the *success probability* of mining for Player i . Let R_i be the random variable for the revenue minus the cost. We call R_i the return of the Nakamoto Reward Process.

Nakamoto Game The Nakamoto Game is a repeated game with the following stage game in finite time interval from $t = 0$ to $t = T$. Let Θ be the Poisson process with $\lambda = \frac{\sum_j M_j/d_j}{D}$. D is a parameter adjusted so that D is close to $\tau \sum_j M_j/d_j$.

1. For all i , Player i chooses balance sheet $\mathcal{B}_i \in \mathcal{O}_i$.
2. Wait a trigger according to Θ .
3. All the Players execute the Nakamoto Reward Process. For all i , Player i with the Mining Assets M_i of $\mathcal{B}_i = (E_i, L_i, M_i, F_i)$ obtains the return R_i .
4. For all i , Player i pays (or receives) interests $r(L_i - F_i)$.

The payoff for each stage game of the Nakamoto Game is $\log(1 + \frac{R_i - r(L_i - F_i)}{E_i})$, and the payoff for the Nakamoto Game is the sum of the payoff of the stage games.

When the Players have the same facility price $d = d_i$, we say that the Players are *homogeneous*. In this case, we normalize the prices without loss of generality so that $d = 1$. We say Player is *static* if it always chooses the fixed balance sheet.

B. Assumption on the Variance of Cost Rate

The Nakamoto Game models the mining cost rate as a constant for each new block arrival, ignoring the timewise variance of the cost. This assumption is realistic for two reasons. First, most of the variance of the return in the one-shot mining comes from the variance from the mining reward and that from the cost rate is minor in practical settings. Second, we are interested in the behavior of Players that remain robust to change of external factors in block arrival timing such as other miner's behavior with possible malicious intentions, the delay of block propagation network, possible forks, and so on.

C. Return from the Nakamoto Reward Process

For a random variable X , the moment generating function of X is defined as $\mathcal{M}_X(u) = \mathbb{E}[e^{uX}]$.

Proposition 1: Given balance sheets $\mathcal{B}_i = (E_i, L_i, M_i, F_i) \in \mathcal{O}_i$ for all i , let R_i be the random variable for the return from the Nakamoto Reward Process for Player i . Then, $\mathcal{M}_{R_i}(u) = \mathcal{M}_{\text{Revenue}}(u)\mathcal{M}_{\text{Cost}}(u)$ for $\mathcal{M}_{\text{Revenue}}(u) = p_i e^{uB} + (1 - p_i)$ and $\mathcal{M}_{\text{Cost}}(u) = e^{-uc_i M_i / d_i}$.

Corollary 1: For the return R_i of the Nakamoto Reward Process with homogeneous Players,

$$\mathbb{E}[R_i] = \frac{BM_i}{\sum_j M_j} - c_i M_i, \text{ and } \mathbb{V}[R_i] = \frac{B^2 M_i M_{-i}}{(\sum_j M_j)^2}$$

for $M_{-i} = \sum_{j \neq i} M_j$.

Corollary 2: Let $Y_i = B/c_i$. Player i 's response in Nakamoto Game satisfies $M_i \leq Y_i - \sum_{j \neq i} M_j$ if $(0, 0, 0, 0) \in \mathcal{O}_i$. In particular $M_i = 0$ if $Y_i \leq \sum_{j \neq i} M_j$.

We call Y_i the break-even hash rate for Player i .

D. Nakamoto Game and Poisson Reward Model

Since the original Bitcoin paper [15], the reward for miners has been modeled using the Poisson process[8], which we call the Poisson reward model.

Poisson Reward Model: A player with hash rate M_i/d_i will receive the revenue B according to the Poisson process with $\lambda_i = \frac{M_i}{Dd_i}$. D is the difficulty parameter, which is adjusted so that $\lambda^{-1} = \tau$ for $\lambda = \frac{H}{D}$, where H is the world hash rate.

We claim that the revenue of the Nakamoto Game with static Players replicates that of the Poisson reward model.

Proposition 2: Suppose static Players play the Nakamoto Game and Players i chooses balance sheet (E_i, L_i, M_i, F_i) for time interval $t = 0$ to $t = T$. Let $\mathcal{M}^i(u)$ be the moment generating function for the sum of the return of player i . Then,

$$\mathcal{M}^i(u) = e^{T\lambda((p_i e^{uB} + 1 - p_i)e^{-uc_i M_i / d_i} - 1)}$$

In particular if $c_i = 0$ then \mathcal{M}^i coincides with the moment generating function of the revenue in the Poisson reward model.

III. DECISION OF ASSET ALLOCATION AND FINANCE

We assume the Players are homogeneous hereafter.

Theorem 1: Suppose Player i plays the Nakamoto Game with $\mathcal{O}_i = \mathcal{O}$. Fix Player j 's balance sheet \mathcal{B}_j for $j \neq i$. Then, there is a unique balance sheet $\mathcal{B}_i^* = (E_i^*, L_i^*, M_i^*, F_i^*) \in \mathcal{O}_i$ that maximizes the expected payoff from the stage game of the Nakamoto Game. Player i maximizes the expected payoff of the Nakamoto Game by continuously choosing \mathcal{B}^* for each stage game. \mathcal{B}^* is determined by $M_i^* = \frac{1}{3} \left(\frac{B}{c_i + r} - M_{-i} \right)$.

Theorem 2: Suppose $(m + n)$ Players play the Nakamoto Game, and let $I = \{1, 2, \dots, m\}$ and $K = \{m + 1, m + 2, \dots, m + n\}$. Suppose that for $i \in I$, Players i tries to maximize the expected payoff of the Nakamoto Game with $\mathcal{O}_i = \mathcal{O}$. For $k \in K$, Player k has only choice of the balance sheet with $\mathcal{O}_k = \{\mathcal{B}_k\}$. Then, there is an equilibrium point $\hat{\mathcal{B}}_i = (\hat{E}_i, \hat{L}_i, \hat{M}_i, \hat{F}_i)$ among Players $i \in I$ in which $\hat{\mathcal{B}}_i$ is determined by

$$\hat{M}_i = \left(\frac{1}{c_i + r} - \frac{1}{m + 2} \sum_{j \in I} \frac{1}{c_j + r} \right) \frac{B}{2} - \frac{Z}{m + 2}$$

for $Z = \sum_{k \in K} M_k$.

A. Optimal Asset Allocation

Proposition 3: Let $W_i = R_i/M_i$, the random variable for the return rate over Mining Assets from the Nakamoto Reward Process with Mining Asset M_i for Player i . For given $\mathcal{B}_i = (E_i, L_i, M_i, F_i)$, the payoff of the stage game of the Nakamoto Game is given by $\log(1 + X_i)$ for $X_i = (1 - f)r + fW_i$, where f is the leverage rate $f = M_i/E_i$.

The expected payoff of the stage game is approximated as

$$\mathbb{E}[\log(1 + X_i)] = r + f(\mathbb{E}[W_i] - r) - \frac{f^2 \mathbb{V}[W_i]}{2} + O(\mathbb{E}[W_i]^2).$$

This approximation formula is obtained by applying the method described in [16].

Let $g_\infty(f) = r + f(\mu - r) - \frac{f^2 \sigma^2}{2}$ for $\mu = \mathbb{E}[W_i]$ and $\sigma^2 = \mathbb{V}[W_i]$. The player would like to maximize $g_\infty(f)$. Solving $g'_\infty(f) = 0$, $g_\infty(f)$ attains the maximum at $f = f^*$ for $f^* = \frac{\mu - r}{\sigma^2}$, and $g_\infty(f^*) = \frac{S^2}{2} + r$ for $S = \frac{\mu - r}{\sigma}$. S is called the Sharpe ratio of W_i . Thus we obtained the following proposition.

Proposition 4: Suppose Player i seeks the optimal balance sheet and fix the balance sheet \mathcal{B}_j for every $j \neq i$. (1) Given Mining Assets \tilde{M} there is a unique balance sheet $\mathcal{B}_i \in \mathcal{O}(\tilde{M})$ that maximizes the expected payoff of the stage game of the Nakamoto Game with the balance sheet chosen from $\mathcal{O}(\tilde{M}) = \{(E, L, M, F) \in \mathcal{O} | M = \tilde{M}\}$. (2) The maximal expected payoff of the stage game of the Nakamoto Game for Player i with balance sheet \mathcal{B}_i is approximated by $S_i^2/2 + r$. S_i is the Sharpe ratio of the return rate of the Single Nakamoto Game, namely $S_i = \frac{\mathbb{E}[W_i] - r}{\sqrt{\mathbb{V}[W_i]}}$.

B. Equilibrium among Players

Suppose $(m+n)$ Players play the Nakamoto Game. Let $I = \{1, 2, \dots, m\}$ and $K = \{m+1, m+2, \dots, m+n\}$. Suppose that for $i \in I$, Player i tries to maximize the expected payoff of the Nakamoto Game with $\mathcal{O}_i = \mathcal{O}$. For $k \in K$, Player k has only choice of the balance sheet with $\mathcal{O}_k = \{\mathcal{B}_k\}$. Let $Z = \sum_{k \in K} M_k$, $H = \sum_{j \in I} M_j + Z$ and $M_{-i} = -M_i + H$ for $i \in I$. The Sharpe ratio of the return rate of Player i in the Single Nakamoto Game is

$$S_i = \frac{1 - \frac{c_i+r}{B}(M_i + M_{-i})}{\sqrt{M_{-i}/M_i}}$$

as Corollary 1 implies $\mathbb{E}[W_i] = \frac{B}{H} - c_i$, $\mathbb{V}[W_i] = \frac{B^2 M_{-i}}{M_i H^2}$.

By solving $\frac{\partial S_i}{\partial M_i} = 0$, we obtain M_i^* that maximizes S_i by $M_i^* = \frac{1}{3}(Y_i' - M_{-i})$ for $Y_i' = \frac{B}{c_i+r}$. By Proposition 4, there exists $\mathcal{B}_i^* = (E_i^*, L_i^*, M_i^*, F_i^*) \in \mathcal{O}(M_i^*)$ that maximizes the expected payoff for choice of balance sheets in $\mathcal{O}(M_i^*)$ for all $i \in I$. Each \mathcal{B}_i^* achieves the maximal expected payoff for any choice of balance sheets in \mathcal{O}_i because it maximizes S_i . This proves Theorem 1.

Using the formula of M_i^* in Theorem 1 we find the equilibrium point $\hat{M}_i = \frac{1}{2}\left(Y_i' - \frac{1}{m+2} \sum_{j \in I} Y_j'\right) - \frac{Z}{m+2}$ from the fixed point of the maximizing condition of S_i for each $i \in I$. Namely, $\hat{M}_i = \frac{1}{3}\left(Y_i' - \hat{M}_{-i}\right)$ for $\hat{M}_{-i} = -\hat{M}_i + \sum_{j \in I} \hat{M}_j + Z$. This concludes Theorem 2. This equation also gives the share of the world hash rate for each Player.

Corollary 3: Let $\hat{H} = \sum_{i \in I} \hat{M}_i + Z$ be the world hash rate at the equilibrium, and $Y_i' = \frac{B}{c_i+r}$. Then,

$$\hat{H} = \frac{1}{m+2} \sum_{i \in I} Y_i' + \frac{2}{m+2} Z, \text{ and } \frac{\hat{M}_i}{\hat{H}} = \frac{1}{2} \left(\frac{Y_i'}{\hat{H}} - 1 \right).$$

In particular, each Player's share of the world hash rate is decided only by the cost rate without explicitly depending on m if the world hash rate at the equilibrium is given.

IV. IMPLICATIONS IN PRACTICE

A. Example

As of February 2020, the real Bitcoin mining environment has the parameters as below.

- 1) The world hash rate is about 1.1×10^8 TH/s.
- 2) Bitcoin price is about 9,500 USD.
- 3) Mining reward is 12.5 BTC.
- 4) The average of time intervals between block arrivals is about 10 minutes.
- 5) An example of the latest mining device is Anteminer S17+. It costs about 2200 USD, including the power supply unit, and it generates about 73 TH/s consuming 2900W power.
- 6) Electric generation charge is about 0.085 USD per kWh.
- 7) US 10-year Treasury Rate is 1.3%. We ignore it because it is small compared with other costs and returns.

Suppose that you are going to start a mining factory that mines 1 out of every 1000 new blocks. If the mining business

is break-even, the mining cost per hash rate (TH) is about $9500 \cdot 12.5 / (1.1 \times 10^8) = 1.1 \times 10^{-3}$ USD. We estimate the cost rate of c_i is 80% of the break-even point.

The world mining assets is $(1.1 \times 10^8) / 73 \cdot 2200 = 3.3 \times 10^9$ USD assuming the homogeneous Players. Your mining assets will be $(3.3 \times 10^9) / (1 - 0.001) \cdot 0.001 = 3.3 \times 10^6$ USD. It is equivalent to about 1500 units of Anteminer S17+. The return rate over your mining assets when mining is successful is $u = (9500 \cdot 12.5 - (1.1 \times 10^8) \cdot 0.001 \cdot (1.1 \times 10^{-3}) \cdot 0.8) / (3.3 \times 10^6) = 3.6 \times 10^{-2}$, and for unsuccessful mining $d = -(1.1 \times 10^8) \cdot 0.001 \cdot (1.1 \times 10^{-3} \cdot 0.8) / (3.3 \times 10^6) = -2.9 \times 10^{-5}$. Applying to $f^* = \frac{up+d(1-p)}{(u^2p+d^2(1-p))-(up+d(1-p))^2}$, f^* is about 5.6.

The optimal log-return rate is about 2.0×10^{-5} per 10 minutes on average. It means the annualized return is about 180%. $f^* = 5.6$ means you should start with about 600,000 USD for Equity, 2,700,000 USD for Liabilities. All the assets are allocated for Mining Assets, 3,300,000 USD.

B. Larger is Not Necessarily Better

Suppose that a player i needs to achieve a given probability p_i of successful mining and chooses the optimal f^* under that constraint. f increases when p_i is small, but decreases for larger p_i and drops to 0 at $p_i = 0.2$: if we add more than 20% of the world hash rate, then the Nakamoto Game becomes unprofitable because the world hash rate exceeds the break-even point. A player will have the motivation to implement a high leverage ratio of over 100. The difficulty in collecting such an amount of Mining Assets may be one of the reasons for forming mining pools, which we discuss in the next section.

C. Mining Pools as the Players

When f^* is high, the Player has an option to work as part of a mining pool. Since the expected simple return rates for miners do not depend on the size of Mining Assets while the variance is smaller as the size of Mining Assets becomes large, the Player can reduce risk by the following methods.

Risk-Sharing Mining Pools Suppose there are a set P of Players who agree that they share the mining reward and dividend it in proportion to the amount of Mining Assets. Let $W_P = \sum_{j \in P} R_j / M_P$ for $M_P = \sum_{j \in P} M_j$, the random variable for return rate for the sum of the returns of the Single Nakamoto Game for Players in P . Then $\mathbb{E}[W_P] = \frac{B}{H} - c_P$, and $\mathbb{V}[W_P] = \frac{B^2(H-M_P)}{M_P H^2}$, for $H = \sum_i M_i$ and $c_P = \sum_{j \in P} \frac{M_j}{M_P} c_j$.

W_P is replicated by a Player with a balance sheet $\mathcal{B} \in \mathcal{O}(M_P)$ for the aggregated Mining Assets M_P with cost rate c_P , and each participating miners are modeled as entities which take the part of the return according to the share of the Mining Assets. This dividend mechanism is modeled out of the Nakamoto Game. The Player with the aggregated Mining Assets is called the Risk-Sharing mining pool.

In practice, Risk-Sharing mining pools were first implemented with a proportional reward policy. However, it is hard

to implement a fair method that verifies each member's contribution to the mining pool's hash rate, and many deceiving methods are proposed, such as hopping attacks. Practically, the most popular implementation at this moment is the pay-per-last- N -share (PPLNS) pools.

Risk-Free Reward Mining Pools Suppose there is a player i who accumulates hash rate by collecting contributions by the external collaborators that receive a risk-free fee.

Suppose Player i offers mining reward of c per hash rate per the average time interval for mining a new block, and collects hash rate $\Phi_i(c)$ performed by the collaborators. Then the pool's success probably of the single Nakamoto Game is $p_i = \frac{\Phi_i(c)}{\Phi_i(c) + M_{-i}}$, so $\Phi_i(c) = \frac{p_i}{1-p_i} M_{-i}$. We can calculate the optimal balance sheet $\mathcal{B}_i(p_i) = (E_i(p_i), L_i(p_i), M_i(p_i), F_i(p_i)) \in \mathcal{O}(\frac{p_i}{1-p_i} M_{-i})$ that produces success probability p_i by applying Proposition 4 to $\tilde{M} = M_i(p_i) = \frac{p_i}{1-p_i} M_{-i}$.

If the mining pool wants the best log-return rate for a given c , then it should prepare Equity $E_i(p_i)$ determined by p_i , but does not need to prepare Liabilities because the Mining Assets are already levered at the optimal ratio. The money prepared as Equity works as the reserve to pay the reward at a cost rate c to collaborators. The return from the mining pool is replicated an ordinary player with balance sheet $\mathcal{B}_i(p_i)$ and cost rate $c_i = c$ but with an extra revenue of $rL_i(p_i)$, the interest of Liabilities at the risk-free rate.

In reality, this type of mining pools is implemented as the pay-per-share (PPS) mining pools. It is a separate interesting topic of how we model the market of tradable hash rates that gives fair Φ_i .

D. Predicting Mining Pools' Shares

According to practitioners, the world break-even price of Bitcoin is about 8,000 USD as of November 2019 [17]. This implies that the world hash rate is estimated to be about 80% – 85% of the break-even hash rate. Corollary 3 implies each of growth-rate mining pools will have about 9%-13% of the hash rate assuming that they have similar cost rates.

If a mining pool is exceptionally cost-efficient by more than about 70% to the other mining pools, then the pool has a reason to occupy more than 50% of the hash rate.

V. CONCLUSIONS

We analyzed how the return from blockchain mining is optimized by dynamic adjustment of the asset allocation for mining resources and of financial structures for mining businesses. We have observed that for each miner, how the optimal share of the hash rate is determined by the mining reward and the mining cost.

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